THE NON-SEPARABLE PROCESS

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THESIS

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ABSTRACT

The work investigates the notion of separability in continuous parameter stochastic processes. It explores the implications of relaxing the separability hypothesis. Various numerical results are obtained for a particular example, the 0-1 Markov process.

TABLE OF CONTENTS

I. INTRODUCTION ---------------------------------- 4

II. SEPARABILITY ---------------------------------- 5

III. THE 0-1 MARKOV PROCESS ------------------------- 16

A. THE SEPARABLE 0-1 MARKOV PROCESS ----------- 17

B. THE NON-SEPARABLE 0-1 MARKOV PROCESS ------- 18

1. Extension of the Measure --------------- 18

2. Inclusion of the 1-Transition Sets ----- 24

3. Arbitrary Assignment of Measures ------- 29

IV. CONCLUSION ------------------------------------ 44

BIBLIOGRAPHY -- 45

INITIAL DISTRIBUTION LIST --------------------------- 46

FORM DD 1473 -- 47

I. INTRODUCTION

Our aim is to investigate peculiarities of non-separable processes. In Chapter II we introduce the general notion of separability for abstract, continuous parameter stochastic processes following Doob's presentation [2]. We then illustrate the concept in action by studying the 0-1 Markov process, to which Chapter III is devoted. We first define the class of measurable sets generated by the finite dimensional measures. Then, following Halmos' presentation of inner and outer measures and extensions [3], we extend the measure in a non-unique fashion to a larger class of sets. Each extension specifies a particular equivalent version of our process, the separable case being one of the collection. Learning that certain constraints were involved, we derive a functional equation later in the chapter, and such a functional equation summarizes all the constraints. We then place the non-separable case under a new light through the use of a "factor function" which we relate to the probabilities within the non-separable process. From this new vantage point we were able to prove that at a particular parameter value either all or none of the sample paths are continuous. We conclude our work with suggestions for further study.

II. SEPARABILITY

In this section we wish to present the concept of separability in stochastic processes that have a continuous parameter set. As will be seen, such processes deal with sets that are not measurable unless certain conditions are placed on the process. The concept of separability gives the required conditions enabling such sets to be measurable. Hopefully, this section will make clear the concept of separability, as well as the problem we wish to present.

Let $\{x_t, t\varepsilon T\}$ be a stochastic process with parameter set T which is any interval of the real line. The random variables x_t are defined on a set Ω. Let $\mathcal{A}$ denote a class of closed sets. Then the process is said to be <u>separable relative to $\mathcal{A}$</u> if there exists a sequence of parameter values $\{t_j\}$ and a subset Λ of Ω of probability zero such that for $A\varepsilon\mathcal{A}$ and any open interval I in T, the ω-sets

$$\{x_t(\omega)\varepsilon A, t\varepsilon I\} \qquad\qquad \{x_{t_j}(\omega)\varepsilon A, t_j\varepsilon I\}$$

differ by at most a subset of Λ. We will use the term "a separable process", acknowledging that it is with respect to $\mathcal{A}$ since the context is clear. Also we will call any sequence $\{t_n\}$ satisfying the conditions of the separability definition, a separability set.

Except in pathological cases the separability set $\{t_n\}$ is dense in T. If we enlarge our separability set by adding a countable number of other parameter values, the new sequence $\{\bar{t}_j\}$ is also a separability set, since $\{t_j\}\subset\{\bar{t}_j\}\subset T$. Conversely, if any countable dense subset is a separability set, we say the process is <u>well-separable</u>.

It was worth questioning whether we could restrict ourselves to open intervals I_r with rational, or infinite, endpoints, and still satisfy the separability definition. The question proves to be true. Consider an interval (a,b) where a and b are irrational. Let $a_n \downarrow a$ and $b_n \uparrow b$, where the a_n's and b_n's are all rational. We observe that for $n>m$, and any $A\varepsilon\mathcal{A}$,

$$M_n \equiv \{x_{t_j}(\omega)\varepsilon A, t_j\varepsilon(a_n,b_n)\} \subseteq M_m \equiv \{x_{t_j}(\omega)\varepsilon A, t_j\varepsilon(a_m,b_m)\}$$

and show that

$$M \equiv \{x_{t_j}(\omega)\varepsilon A, t_j\varepsilon(a,b)\} = \bigcap_{n=1}^{\infty} M_n .$$

Certainly $M \subseteq \bigcap_{n=1}^{\infty} M_n$. So take an $\omega\varepsilon \bigcap_{n=1}^{\infty} M_n$. If $\omega\notin M$, then there is a $\bar{t}_j\varepsilon(a,b)$ such that $x_{\bar{t}_j}(\omega)\notin A$. But there exists an $\bar{n}$ such that $\bar{t}_j\varepsilon(a_{\bar{n}},b_{\bar{n}})$, and $\omega\varepsilon M_{\bar{n}}$ which gives us a contradiction. Thus $\omega\varepsilon M$. Hence, $M = \bigcap_{n=1}^{\infty} M_n$. Therefore in our definition of separability, it is sufficient to restrict the condition to open intervals of rational, or infinite, endpoints.

Looking at the two ω-sets in our definition of separability, we could express the sets in the form,

$$(1) \qquad \{x_t(\omega)\varepsilon A, t\varepsilon I\} = \bigcap_{t\varepsilon I} \{x_t(\omega)\varepsilon A\} \qquad \text{and}$$

$$(2) \qquad \{x_{t_j}(\omega)\varepsilon A, t_j\varepsilon I\} = \bigcap_{t_j\varepsilon I} \{x_{t_j}\varepsilon A\} .$$

The ω-set (2) is the intersection of a countable number of measurable sets, hence, the intersection is measurable. Though each ω-set in the intersection of Equation (1) is measurable, this is not enough to show that the intersection is measurable, since the intersection is over an uncountable number of sets. However, because of the separable condition, ω-set (1) has the same probability as ω-set (2). This subject of countable or uncountable operations seemed to be the main problem in the non-separable processes we considered; once we leave the countable realm, we become very dependent on the concept of separability.

Now consider the operations of greatest lower bound, glb, and least upper bound, lub, and suppose our process is separable. Choose an $\bar{\omega}$ not in the null set Λ. Then for a separability set $\{t_j\}$, and any open interval I, we can find an a such that

$$a = \operatorname*{glb}_{t_j\varepsilon I} x_{t_j}(\bar{\omega}), \qquad -\infty \leq a.$$

If $b = \operatorname*{glb}_{t\varepsilon I} x_t(\bar{\omega})$, we wish to show that a = b using the

separability definition. Certainly $b \leq a$. Assume $b < a$, then consider the ω-sets,

$$\{x_t(\omega)\varepsilon[a,\infty],t\varepsilon I\} \quad \text{and} \quad \{x_{t_j}(\omega)\varepsilon[a,\infty],t_j\varepsilon I\} .$$

By the definition of separability, the two ω-sets differ only by a subset of Λ, and since $\overline{\omega}$ is not in the first ω-set, but is an element of the second, $\overline{\omega}$ must be in Λ, which is a contradiction. Therefore, $a = b$, or, more specifically, $\underset{t\varepsilon I}{\text{glb}}\ x_t(\omega) = \underset{t_j\varepsilon I}{\text{glb}}\ x_{t_j}(\omega)$ for all $\omega \notin \Lambda$. In a similar fashion we could show that $\underset{t\varepsilon I}{\text{lub}}\ x_t(\omega) = \underset{t_j\varepsilon I}{\text{lub}}\ x_{t_j}(\omega)$ for all $\omega \notin \Lambda$. In particular, if $\omega \notin \Lambda$ and if $\overline{t} \varepsilon I$, then

$\underset{t_j\varepsilon I}{\text{glb}}\ x_{t_j}(\omega) = \underset{t\varepsilon I}{\text{glb}}\ x_t(\omega) \leq x_{\overline{t}}(\omega)$, and

$x_{\overline{t}}(\omega) \leq \underset{t\varepsilon I}{\text{lub}}\ x_t(\omega) = \underset{t_j\varepsilon I}{\text{lub}}\ x_{t_j}(\omega)$. Thus,

$$(3) \qquad \lim_{n\to\infty} \underset{|t_j-\overline{t}|<\frac{1}{n}}{\text{glb}} x_{t_j}(\omega) \leq x_{\overline{t}}(\omega) \leq \lim_{n\to\infty} \underset{|t_j-\overline{t}|<\frac{1}{n}}{\text{lub}} x_{t_j}(\omega), \quad \omega \notin \Lambda .$$

A classical theorem concerning separability utilizes (3), stating that if a process is separable, and a sequence $\{t_j\}$ satisfies (3) for every $\overline{t} \varepsilon T$, then the sequence is a separability set. If we place the further condition that if for every $\overline{t} \varepsilon T$ we have

$$(4) \qquad \underset{t\to\bar{t}}{\mathrm{p\,lim}}\; x_t = x_{\bar{t}} \qquad \text{(convergence in probability)},$$

then any countable dense subset of T is a separability set. This is true, for if we take any countable dense subset of T, it satisfies condition (4) by hypothesis. Thus, this subset satisfies (3). Therefore the process is well-separable.

Showing that $\underset{t_j \varepsilon I}{\mathrm{glb}}\; x_{t_j}(\omega) = \underset{t \varepsilon I}{\mathrm{glb}}\; x_t(\omega)$ and $\underset{t_j \varepsilon I}{\mathrm{lub}}\; x_{t_j}(\omega) = \underset{t \varepsilon I}{\mathrm{lub}}\; x_t(\omega)$ for all I open, $\omega \notin \Lambda$, in the separable process, we obtain another natural result: For $\omega \notin \Lambda$

$$(5) \qquad \liminf_{t_j\to\bar{t}} x_{t_j}(\omega) = \lim_{n\to\infty} \underset{|t_j-\bar{t}|<\frac{1}{n}}{\mathrm{glb}}\; x_{t_j}(\omega)$$

$$= \lim_{n\to\infty} \underset{|t_j-\bar{t}|<\frac{1}{n}}{\mathrm{glb}}\; x_t(\omega) = \liminf_{t\to\bar{t}} x_t(\omega)$$

and

$$(6) \qquad \limsup_{t_j\to\bar{t}} x_{t_j}(\omega) = \lim_{n\to\infty} \underset{|t_j-\bar{t}|<\frac{1}{n}}{\mathrm{lub}}\; x_{t_j}(\omega)$$

$$= \lim_{n\to\infty} \underset{|t-\bar{t}|<\frac{1}{n}}{\mathrm{lub}}\; x_t(\omega) = \limsup_{t\to\bar{t}} x_t(\omega).$$

Thus, in the separable process, $\liminf_{t\to\bar{t}} x_t(\omega)$ and $\limsup_{t\to\bar{t}} x_t(\omega)$ are random variables. With this result, the

probability of events concerning limits, strong continuity, etc., may be expressed in terms of probabilities of lim inf's and lim sup's, and, hence, is determinable.

Now there is the question of whether or not any stochastic process can be made separable, without changing the character of the process to any appreciable degree. In other words we are speaking of equivalent processes; a process $\{x_t, t\varepsilon T\}$ is <u>equivalent</u> to the process $\{\tilde{x}_t, t\varepsilon T\}$ if and only if $P\{x_t(\omega) = \tilde{x}_t(\omega)\} = 1$, for each $t \varepsilon T$. So restating the question, for each stochastic process $\{x_t(\omega), t\varepsilon T\}$ does there exist an equivalent process $\{\tilde{x}_t(\omega), t\varepsilon T\}$ such that the $\tilde{x}$-process is separable? We will show that the statement is true.

As is necessary in the definition of separability, we must find a separability set and a null set Λ. Our separability set will be formed by construction, and then from this separability set we will describe parts of our null set.

First, we show that for each closed set A, we can find a sequence $t_1, t_2, \ldots$ such that

(7) $$P\{x_{t_n}(\omega)\varepsilon A, n\geq 1; x_t(\omega)\notin A\} = 0 \quad \text{for each } t \varepsilon T.$$

So take a closed set A. Now choose any finite sequence $t_1, \ldots, t_k$ of parameter values. Define

$$G_{j,t} = \{x_{t_n}(\omega)\varepsilon A, n\leq j; x_t(\omega)\notin A\}, \quad \text{and}$$

$$\rho_j = \operatorname*{lub}_{t\varepsilon T} P(G_{j,t})$$

We note that for a fixed t, the $G_{j,t}$'s form a decreasing sequence of sets, implying that $\rho_1 \geq \rho_2 \geq \cdots$ If $\rho_k = 0$, then we have found the desired sequence for the particular closed set. If $\rho_k > 0$, by the definition of lub, there exists a τ such that $P(G_{k,\tau}) \geq \rho_k(1 - \frac{1}{k})$, and we let $t_{k+1} = \tau$. Then, by induction, we form an infinite sequence $t_1, t_2, \ldots$ Assume that $\rho_k > 0$, for all k. For each t, we have

$$(8) \qquad P\{x_{t_n}(\omega)\varepsilon A, n\geq 1; x_t(\omega)\notin A\} \leq \lim_{k\to\infty} \rho_k.$$

Consider the ω-sets,

$$H_k = \{x_{t_n}(\omega)\varepsilon A, n\leq k; x_{t_{k+1}}\notin A\}.$$

Since the H_k's are disjoint, the probability of their union equals the sum of their probabilities, implying that $\lim_{k\to\infty} P(H_k) = 0$. But by definition, $P(H_k) \geq \rho_k(1 - \frac{1}{k})$; thus, $\lim_{k\to\infty} \rho_k = 0$. From (8),

$$P\{x_{t_n}(\omega)\varepsilon A, n\geq 1; x_t(\omega)\notin A\} = 0, \qquad \text{for each } t \varepsilon T,$$

and so we have found the desired sequence for a particular closed set A.

Now let $\mathcal{A}_o$ be the class of sets which are finite unions of closed intervals with rational or infinite endpoints. If $\mathcal{A}$ is the class of sets which are intersections of sequences of sets in $\mathcal{A}_o$, then $\mathcal{A}$ includes all the closed sets. By the preceding paragraph, for each $A \in \mathcal{A}_o$ there corresponds a particular parameter sequence such that (7) is true. Since there is a countable number of closed sets in $\mathcal{A}_o$, we let $\{t_n\}$ be the union of all the corresponding sequences. Let

$$\Lambda_t(A) = \{x_{t_n}(\omega)\varepsilon A, n\geq 1; x_t(\omega)\notin A\} \ , \quad \text{and}$$

$$\Lambda_t = \bigcup_{A\varepsilon\mathcal{A}_o} \Lambda_t(A) \ .$$

We note that $P(\Lambda_t) = 0$, since there are only a countable number of $\Lambda_t(A)$'s to consider, each of which has probability 0. Now if $\bar{A} \varepsilon \mathcal{A}$, and $A_o \varepsilon \mathcal{A}_o$ such that $\bar{A} \subset A_o$, then

$$(9) \qquad \{x_{t_n}(\omega)\varepsilon\bar{A}, n\geq 1; x_t(\omega)\notin \mathring{A}\} \subset \{x_{t_n}(\omega)\varepsilon A_o, n\geq 1; x_t(\omega)\notin \mathring{A}\} \subset \Lambda_t .$$

The implication then follows, since each $\mathcal{A}$-set is an intersection of $\mathcal{A}_o$-sets.

Now let I be an open interval with rational or infinite endpoints, with $I \subset T$, and then restrict our process to the

process $\{x_t, t\varepsilon I\}$. For this restriction, we let $\{t_n\}_I = \{t_n\} \cap I$. Then by our previous results, there exists an ω-set $\Lambda_{t,I}$, such that

$$P(\Lambda_{t,I}) = 0 \quad \text{for each } t \varepsilon I, \quad \text{and}$$

$$\{x_s(\omega)\varepsilon A, s\varepsilon\{t_n\}_I; x_t(\omega)\notin A\} \subset \Lambda_{t,I}, \quad \text{for each } A \varepsilon \mathcal{A}.$$

Let

$$S = \bigcup_{I\varepsilon I_r} \{t_n\}_I, \quad \Lambda_t = \bigcup_{I\varepsilon I_r} \Lambda_{t,I} \quad \text{where } I_r \text{ is}$$

the class of all open intervals in T with rational or infinite endpoints, implying I_r is countable class of open intervals. For a fixed ω, we define $A(I,\omega)$ to be the closure of the set of values of $x_s(\omega)$ where s varies in $I \cap S$. The values of $\pm\infty$ may be in $A(I,\omega)$. This set is closed and non-empty, and we note that

$$x_t(\omega)\varepsilon A(I,\omega) \quad \text{if} \quad t\varepsilon I, \omega\notin\Lambda_t.$$

Defining $A(t,\omega) = \bigcap_{I\varepsilon t} A(I,\omega)$, we know that $A(t,\omega)$ is closed, and since we have made the class of $A(I,\omega)$'s compact by including the values $\pm\infty$, $A(t,\omega)$ is non-empty. So

$$x_t(\omega)\varepsilon A(t,\omega) \quad \text{if} \quad t\varepsilon T, \omega\notin\Lambda_t.$$

Thus, we define our $\tilde{x}$-process by,

$$\tilde{x}_t(\omega) = \begin{cases} x_t(\omega) & \text{if} \quad t \in S \\ x_t(\omega) & \text{if} \quad t \notin S \quad \omega \notin \Lambda_t \\ \text{any value in } \Lambda(t,\omega) & \text{if } t \notin S,\ \omega \in \Lambda_t. \end{cases}$$

We now want to show that the $\tilde{x}$-process is separable, and equivalent to the original x-process. Certainly $P\{x_t(\omega) = \tilde{x}_t(\omega)\} = 1$, implying that the two processes are equivalent. Now let $A \in \mathcal{A}$, and suppose that $I \in I_r$ and $\bar{\omega}$ is such that $\tilde{x}_s(\bar{\omega}) \in A$ for all $s \in I \cap S$. Hence, by definition $A(I,\omega) \subset A$. So if $t \in I$, then by our construction of $\tilde{x}$,

$$\tilde{x}_t(\omega) = x_t(\omega) \in \begin{cases} A(I,\omega) \subset A & \text{if} \quad t \in S, \quad \text{or} \quad t \notin S, \omega \notin \Lambda_t \\ A(t,\omega) \subset A(I,\omega) \subset A & \text{if} \quad t \notin S, \omega \in \Lambda_t. \end{cases}$$

Thus,

(10) $\{\tilde{x}_s(\omega) \in A, s \in I \cap S\} = \{\tilde{x}_t(\omega) \in A, t \in I\}$.

By a previous argument, we know that we can restrict the definition of separability to open intervals with rational or infinite endpoints. Therefore our $\tilde{x}$-process is separable, and thus, every stocahstic process has an equivalent, separable process.

It is important to note some characteristics of this constructed separable process before leaving this section. Though it may be too strong to say that the null set Λ of this process is the void set, it is, nonetheless, true that for all open intervals I in T and A in $\mathcal{A}$, the two ω-sets

$$\{\tilde{x}_s(\omega)\varepsilon A, s\varepsilon I \cap S\} \quad \text{and} \quad \{\tilde{x}_t(\omega)\varepsilon A, t\varepsilon I\}$$

differ by a void set. The construction has eliminated from the x-process all sample paths, characterized by an ω, which behave, in some sense, badly. There are possibly an uncountable number of separable processes equivalent to the x-process, but this $\tilde{x}$-process is the best one can hope to have, in the sense that every sample path behaves well. Thus, this $\tilde{x}$-process represents one extreme of the class of all processes equivalent to the x-process. It is fair then to ask what process represents the other extreme. Because of the abstractness of our sample paths, we delay this question until a later section, where we consider a particular process in which the sample paths are easier to picture.

III. THE 0-1 MARKOV PROCESS

In considering the separability concept, the continuous parameter process we will study is the 0-1 Markov process $\{x_t(\omega), t\varepsilon[0,t_o]\}$, where $t_o < \infty$. In such a process, whether separable or not, certain finite dimensional probability properties are true. Define $p_{ij}(s,t)$ to be the transition probability; that is,

$$p_{ij}(s,t) = P\{x_t(\omega) = j \mid x_s(\omega)=i\}, \quad \text{where } 0 \le s < t \le t_o,$$
$$i,j\varepsilon\{0,1\},$$
$$\text{and} \quad P\{x_s(\omega)=i\}>0.$$

For our process, the transition probabilities have the stationary property that $p_{ij}(s,t)$ depends only on the difference $t - s$, and we may write the transition probability as $p_{ij}(t-s)$. Further, these transition functions can be derived and are of the form:

$$p_{oo}(t) = a + be^{-t} \qquad p_{o1}(t) = b - be^{-t}$$

$$p_{11}(t) = b + ae^{-t} \qquad p_{10}(t) = a - ae^{-t}$$

where $a,b > 0$, and $a + b = 1$.

A. THE SEPARABLE 0-1 MARKOV PROCESS

Now let us suppose that our process is separable. It can be shown that

$$P\{x_t(\omega)\equiv i, \hat{t}\leq t\leq \hat{t}+\alpha \mid x_{\hat{t}}(\omega)=i\} = e^{-q_i\alpha}$$

where $q_i = \begin{cases} b & \text{if} \quad i = 0 \\ a & \text{if} \quad i = 1\ . \end{cases}$

We know also that each sample path in which there exists an instantaneous jump from one state to another has probability 0. In particular, sample paths of the form,

$$\{x_t(\omega)=0, t\varepsilon[0,\hat{t}]; x_t(\omega)=1, t\varepsilon(\hat{t},t_o], \mid x_o(\omega)=0\}$$

have probability 0. However if we take the probability of the collection of all such sample paths (that is, the probability of exactly one transition in the interval $[0,t_o]$), we obtain $\frac{b}{b-a}(e^{-at_o} - e^{-bt_o})$. If $b = a = \frac{1}{2}$, the probability is derived by taking the limit as b and a both approach ½, and the probability is $\frac{1}{2}t_o e^{-\frac{1}{2}t_o}$. These probabilities are conditioned on the event that $x_o(\omega) = 0$. If we condition them on the event that $x_o(\omega) = 1$, the probability of exactly one transition in the same interval is $\frac{a}{a-b}(e^{-bt_o} - e^{-at_o})$, and again if $a = b = \frac{1}{2}$, we get $\frac{1}{2}t_o e^{-\frac{1}{2}t_o}$. As will be seen later in this section, these

probabilities are bounds for the arbitrary probabilities that may result from the same events in the non-separable processes. Other "separable" probabilities for different events could also be calculated, but those mentioned above will suffice for our discussion of the non-separable process.

B. THE NON-SEPARABLE 0-1 MARKOV PROCESS

1. Extension of the Measure

The finite dimensional probabilities yield a measure for events involving a countable number of parameter values. The collection of all these events will be the σ-algebra S, with the probability measure μ. We now wish to prove and apply the following extension theorem:

Theorem: If $M \varepsilon S$, there exists a (non-unique) probability measure $\tilde{\mu}$, an extension of μ, defined on the σ-algebra $\tilde{S}$ generated by the sets in S and the set M.

Proof: We first show that $\mathcal{A} = \{(E \cap M) \cup (F \cap M') : E, F \varepsilon S\}$ is the σ-algebra $\tilde{S}$ that we are looking for in the theorem. Since $X, \phi \varepsilon S$, we have that $M = (X \cap M) \cup (\phi \cap M') \varepsilon \mathcal{A}$. Also, for all $E \varepsilon S$, it follows that $E = (E \cap M) \cup (E \cap M') \varepsilon \mathcal{A}$, implying that $S \subset \mathcal{A}$. Thus, $\mathcal{A}$ is in a sense generated by the sets in S and the set M. Now we must show that $\mathcal{A}$ is a σ-algebra.

Let $A_1, A_2, \ldots \varepsilon \mathcal{A}$, implying there exist $E_1, E_2, \ldots, F_1, F_2, \ldots \varepsilon S$, such that

$A_1 = (E_1 \cap M) \cup (F_1 \cap M')$, $A_2 = (E_2 \cap M) \cup (F_2 \cap M')$, ... Since

$\bigcup\limits_{n=1}^{\infty} E_n \varepsilon S$ and $\bigcup\limits_{n=1}^{\infty} F_n \varepsilon S$, we have that

$\bigcup\limits_{n=1}^{\infty} A_n = [(\bigcup\limits_{n=1}^{\infty} E_n) \cap M] \cup [(\bigcup\limits_{n=1}^{\infty} F_n) \cap M'] \varepsilon \mathcal{A}$. Also if

$A = (E \cap M) \cup (F \cap M') \varepsilon \mathcal{A}$, then

$$A' = [(E \cap M) \cup (F \cap M')]' = (E' \cup M') \cap (F' \cup M) = (E' \cap F') \cup (E' \cap M) \cup (F' \cap M')$$

$$= (E' \cap M) \cup (F' \cap M) \varepsilon \mathcal{A}.$$ Therefore, $\mathcal{A}$ is a σ-algebra.

Let m_* and m^* denote the inner and outer measure, respectively, with respect to measure μ. If $m_*(M) = a > 0$, then there exists for each n a $G_n \varepsilon S$ such that $G_n \subset M$ and $a - \frac{1}{n} \leq \mu(G_n) \leq a$. Let $G = \bigcup\limits_{n=1}^{\infty} G_n$. Certainly $G \subset M$, and $G \varepsilon S$. Further, by the definition of inner measure, $m_*(M - G) = 0$. Now if $m^*(M) = b > 0$, in a similar fashion we can find a sequence of H_n's in S such that if $H = \bigcap\limits_{n=1}^{\infty} H_n$, then $M \subset H$, and $m_*(H - M) = 0$. Thus, we have found $G, H \varepsilon S$ such that $G \subset M \subset H$ and $\mu(G) = m_*(M)$ and $m^*(M) = \mu(H)$.

Now let $D = H \cap G'$, and let $A = (E \cap M) \cup (F \cap M') \varepsilon \mathcal{A}$. Then $A \cap D' = (E \cap M \cap D') \cup (F \cap M' \cap D') = (E \cap G) \cup (F \cap H') \varepsilon S$. Thus, for all $A \varepsilon \mathcal{A}$, $A \cap D' \varepsilon S$. We may rewrite $m_*(M-G) = m_*(D \cap M)$ and $m_*(H-M) = m_*(D \cap M')$. So

$$m^*(D \cap M) = m^*(D \cap M) + m_*(D \cap M') = \mu(D) = m^*(D \cap M') + m_*(D \cap M)$$
$$= m^*(D \cap M').$$

Let $A_1 = (E_1 \cap M) \cup (F_1 \cap M')$ and $A_2 = (E_2 \cap M) \cup (F_2 \cap M')$, with $A_1, A_2 \varepsilon \mathcal{A}$. Suppose $A_1 \cap D = A_2 \cap D$. Then it must follow that $(E_1 \cap M \cap D) = (E_2 \cap M \cap D)$ and $(F_1 \cap M' \cap D) = (F_2 \cap M' \cap D)$. If $(E_1 \cap D)$ and $(E_2 \cap D)$ disagree, then their symmetric difference,

$(E_1 \cap D) \Delta (E_2 \cap D)$, is a subset of M'. Thus, $(E_1 \cap D) \Delta (E_2 \cap D) \subset D \cap M'$. Hence, $\mu((E_1 \cap D) \Delta (E_2 \cap D)) \leq m_*(M' \cap D) = 0$. Therefore, $\mu(E_1 \cap D) = \mu(E_2 \cap D)$. In the same fashion we can show that $\mu(F_1 \cap D) = \mu(F_2 \cap D)$.

Now we define our extended measure $\tilde{\mu}$ on the σ-algebra $\mathcal{A}$. Let $\alpha, \beta \geq 0$, with $\alpha + \beta = 1$. Then for $A = (E \cap M) \cup (F \cap M') \in \mathcal{A}$, $\tilde{\mu}$ is defined by,

$$\tilde{\mu}((E \cap M) \cup (F \cap M')) = \mu([(E \cap M) \cup (F \cap M')] \cap D') + \alpha\mu(E \cap D)$$

$$+ \beta\mu(F \cap D).$$

By the preceding paragraph, such a measure is well-defined, and certainly for all $A \in \mathcal{A}$, $\tilde{\mu}(A) \geq 0$. If $A = \phi$, then

$$\tilde{\mu}(\phi) = \mu([(\phi \cap M) \cup (\phi \cap M')] \cap D') + \alpha\mu(\phi \cap D) + \beta\mu(\phi \cap D)$$

$$= \mu(\phi) + \alpha\mu(\phi) + \beta\mu(\phi) = 0.$$

Now let $\{A_n\}$ be a disjoint sequence in $\mathcal{A}$, where $A_n = (E_n \cap M)\ (F_n \cap M')$. Then,

$$\tilde{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) = \tilde{\mu}\left(\left(\bigcup_{n=1}^{\infty} E_n\ M\right) \cup \left(\bigcup_{n=1}^{\infty} F_n\ M'\right)\right)$$

$$= \mu\left(\left[\left(\bigcup_{n=1}^{\infty} E_n \cap M\right) \cup \left(\bigcup_{n=1}^{\infty} F_n \cap M'\right)\right] \cap D'\right) + \alpha\mu\left(\bigcup_{n=1}^{\infty} E_n \cap D\right) + \beta\mu\left(\bigcup_{n=1}^{\infty} F_n \cap D\right)$$

$$= \mu\left(\bigcup_{n=1}^{\infty} [(E_n \cap M \cap D') \cup (F_n \cap M' \cap D')]\right) + \sum_{n=1}^{\infty} \alpha\mu(E_n \cap D) + \sum_{n=1}^{\infty} \beta\mu(F_n \cap D)$$

$$= \sum_{n=1}^{\infty} (\mu[((E_n \cap M) \cup (F_n \cap M')) \cap D']) + \sum_{n=1}^{\infty} \alpha\mu(E_n \cap D) + \sum_{n=1}^{\infty} \beta\mu(F_n \cap D)$$

$$= \sum_{n=1}^{\infty} (\mu[((E_n \cap M) \cup (F_n \cap M')) \cap D'] + \alpha\mu(E_n \cap D) + \beta\mu(F_n \cap D))$$

$$= \sum_{n=1}^{\infty} \tilde{\mu}(A_n).$$

Thus, $\tilde{\mu}$ is a measure on $\mathcal{A}$. Further, since

$(E \cap M) \cup (E \cap M') = E \in S$,

$\tilde{\mu}(E) = \mu(E \cap D') + (\alpha+\beta)(\mu(E \cap D) = \mu(E \cap D') + \mu(E \cap D) = \mu(E)$.

Hence, $\tilde{\mu} = \mu$ on S. Also, since

$\tilde{\mu}(M) = \mu(M \cap D') + \alpha\mu(D) = \mu(G) + \alpha\mu(H \cap G') = \mu_*(M) + \alpha\mu(H \cap G')$.

If $\alpha = 1$, then $\tilde{\mu}(M) = \mu(G) + \mu(H \cap G') = \mu(H) = \mu^*(M)$.

Therefore, $\mu_*(M) \leq \tilde{\mu}(M) \leq \mu^*(M)$, depending on α (and β).

Setting $S = \mathcal{A}$, we have proved the theorem.

Now suppose we have a countable sequence of sets, M_1, M_2, ..., not in S. Let $S_n(M_{n+1})$ be the extended σ-algebra, S_{n+1}, generated by the sets in the σ-algebra S_n and the set M_{n+1}. It follows that,

$$S \subset S_1 \equiv S(M_1) \subset S_2 \equiv S_1(M_2) \subset \ldots$$

Let $\mathcal{F} = \bigcup_{n=1}^{\infty} S_n$. If A and B are in $\mathcal{F}$, then there exist a j and k such that $A \in S_j$ and $B \in S_k$. Without loss of generality, we can assume $j \geq k$. Then $B \in S_j$. Thus, $A \cup B \in S_j \subset \mathcal{F}$. Certainly if $C \in \mathcal{F}$, then $C \in S_\ell$ for some ℓ, implying that $C' \in S_\ell \subset \mathcal{F}$. Therefore, $\mathcal{F}$ is an algebra.

The measure we apply to $\mathcal{F}$ is μ_s defined by

$\mu_s(A) = \mu_k(A)$ where $A \in S_k$, and μ_k is the measure on S_k.

We now apply the Caratheodory Extension Theorem:

Theorem: Let μ be a measure on an algebra $\mathcal{A}$, and μ^* the outer measure induced by μ. Then the restriction $\bar{\mu}$ of μ^* to the μ^*-measurable sets is an extension of μ to a σ-algebra containing $\mathcal{A}$. If μ is finite, so is $\bar{\mu}$. If μ is σ-finite, then $\bar{\mu}$ is the only measure on the smallest σ-algebra containing $\mathcal{A}$ which is an extension of μ.

We relate this extension by considering the M-sets associated with intervals in $[0,t_o]$. For any open interval, (r,s) where r and s are rational, we define,

$$M_{1,(r,s)} \equiv \{x_t(\omega)=1 \text{ on } (r,s)\} , \quad \text{and}$$

$$\mathcal{M}_1 \equiv \{M_{1,(r,s)} | r \text{ and } s \text{ are rational}\} .$$

Defining $\mathcal{M}_o$ similarly, we let $\mathcal{M} = \mathcal{M}_o \cup \mathcal{M}_1$.

Let S be the σ-algebra of all sets which are measurable by our finite dimensional probabilities. Since the class $\mathcal{M}$ is countable, by the two extension theorems above there is a σ-algebra $\mathcal{A}$ containing $\mathcal{M}$ and S. Thus, the $\mathcal{M}$-sets are now measurable. Also this extension $\mathcal{A}$ has produced additional M-sets, namely those associated with open intervals with irrational endpoints, so these are also measurable. Such an M-set is generated by the intersection of a countable number $\mathcal{M}$-sets. It is crucial to realize that if $r_n \downarrow r$ and $s_n \uparrow s$, then
$\lim_{n\to\infty} M_{1,(r_n,s_n)} = \bigcap_{n=1}^{\infty} M_{1,(r_n,s_n)} = M_{1,(r,s)}$. Also,

$$P(M_{1,(r,s)} - M_{1,[r,s]}) =$$

$$= P([\{x_r(\omega)=0\} \cap M_{1,(r,s)} \cap \{x_s(\omega)=1\}] \cup [\{x_r(\omega)=1\} \cap M_{1,(r,s)}$$

$$\cap \{x_s(\omega)=0\}] \cup [\{x_r(\omega)=x_s(\omega)=0\} \cap M_{1,(r,s)}])$$

$= 0$, by properties of the Markov process.

Hence, $P(M_{1,(r,s)}) = P(M_{1,[r,s]})$. Similarly, the same result is obtained for any M_0-set.

Let m be the measure defined on S, and m_* and m^* be the inner and outer measure, respectively, with respect to the sets in S. For any open interval I, $m_*(M_{1,I}) = 0$, since the only set in S contained in $M_{1,I}$ is the void set. It is to be noted also that in the separable case, $P(M_{1,I}) = m^*(M_{1,I})$. However in the non-separable case, by the first extension theorem, the most we have for the probability of $M_{1,I}$ is,

$$0 = m_*(M_{1,I}) \leq P(M_{1,I}) \leq m^*(M_{1,I}) = \hat{P}(M_{1,I})$$

where $\hat{P}$ denotes the probability in the separable case.

Suppose $P(M_{1,(r,s)}) = \theta\hat{P}(M_{1,(r,s)})$. It is easily shown that $P(M_{1,(r,s)}|x_r(\omega) = 1) = \theta\hat{P}(M_{1,(r,s)}|x_r(\omega) = 1)$. Therefore, hereafter when we evaluate the probability of an M-set, we are evaluating the conditional probability of the set, conditioned on the initial value. Thus,

$$\hat{P}(M_{i,I}) = e^{-q_i|I|}, \quad \text{where } q_i = \begin{cases} b & \text{if} \quad i = 0 \\ a & \text{if} \quad i = 1. \end{cases}$$

Also, henceforth, $\mathcal{M}$ will be the class of all $M_{j,I}$ regardless of the type of interval I.

2. <u>Inclusion of the 1-Transition Sets</u>

From the preceding section, we have shown that all sample paths which are constant (0 or 1) on an open interval

are measurable in $\mathcal{J}$, with the measure being somewhat arbitrary. (We will see in later sections that there are other restrictions on the measure.) In this section we wish to show that all sample paths making one transition between parameter values 0 and t_o can be described by these $\mathcal{M}$-sets, hence, are also measurable.

We use the following notation:

$$M^{2^n}_{o,k} \equiv \{x_t(\omega) = 0 \quad \text{on } [0, \frac{kt_o}{2^n})\}$$

$$M^{2^n}_{1,k+1} \equiv \{x_t(\omega) = 1 \quad \text{on } (\frac{(k+1)t_o}{2^n}, t_o]\}$$

$$M_1^{k,2^n} \equiv M^{2^n}_{0,k} \cap M^{2^n}_{1,k+1}$$

$$M_1^{2^n} = \bigcup_{k=1}^{2^n-2} M_1^{k,2^n}$$

We now show that the $\lim\limits_{n\to\infty} M_1^{2^n}$ exists. It is sufficient to show that $\overline{\lim\limits_{n\to\infty}} M_1^{2^n} \subseteq \underline{\lim\limits_{n\to\infty}} M_1^{2^n}$. Let $\omega \varepsilon \overline{\lim\limits_{n\to\infty}} M_1^{2^n}$. Then $\omega \varepsilon M_1^{2^{\bar{n}}}$ for some $\bar{n}$. Further, there exists an $n' > \bar{n}$ such that $\omega \varepsilon M_1^{2^{n'}}$. Thus, $\omega \varepsilon M_1^{\bar{k},2^{\bar{n}}}$ and $\omega \varepsilon M_1^{k',2^{n'}}$ for some $\bar{k}$ and k', with the added condition that

$$\frac{\bar{k}t_o}{2^{\bar{n}}} \leq \frac{k't_o}{2^{n'}} < \frac{(k'+1)t_o}{2^{n'}} \leq \frac{(\bar{k}+1)t_o}{2^{\bar{n}}} .$$

Then we can find a k such that

$$\frac{\bar{k}t_o}{2^{\bar{n}}} \leq \frac{kt_o}{2^{\bar{n}+1}} \leq \frac{k't_o}{2^{n'}} < \frac{(k'+1)t_o}{2^{n'}} \leq \frac{(k+1)t_o}{2^{\bar{n}+1}} \leq \frac{(\bar{k}+1)t_o}{2^{\bar{n}}} .$$

Thus, $\omega \in M_1^{k,2^{\bar{n}+1}} \subset M_1^{2^{\bar{n}+1}}$. Hence, by induction, $\omega \in M_1^{2^n}$ for all $n \geq \bar{n}$. So $\omega \in \underline{\lim}_{n\to\infty} M_1^{2^n}$, implying that $\overline{\lim}_{n\to\infty} M_1^{2^n} \subseteq \underline{\lim}_{n\to\infty} M_1^{2^n}$. Therefore $\lim_{n\to\infty} M_1^{2^n}$ exists.

Let $A = \{x_t(\omega)=0 \text{ on } [0,\hat{t}), x_t(\omega)=1 \text{ on } (\hat{t},t_o], \text{ for some } \hat{t}\}$. We note that A is the collection of all sample paths which makes only one transition to 1 in the interval $[0,t_o]$. We now want to show that $\lim_{n\to\infty} M_1^{2^n} = A$.

For $\omega \in A$ and $\hat{t}(\omega)$, we can find a positive integer N such that for each $n > N$, there is a k_n satisfying

$$\frac{k_n t_o}{2^n} \leq \hat{t} \leq \frac{(k_n+1)t_o}{2^n} .$$

Thus, $\omega \in M_1^{2^n}$ for all $n > N$. Hence, $\omega \in \lim_{n\to\infty} M_1^{2^n}$, implying that $A \subseteq \lim_{n\to\infty} M_1^{2^n}$.

For $\omega \in \lim_{n\to\infty} M_1^{2^n}$, there is a positive integer N such that for each $n > N$, there is at least one k_n such that $\omega \in M_1^{k_n,2^n}$. In particular we can find a sequence of $\mathcal{M}$-sets,

$$M_1^{k_n,2^n} \supset M_1^{k_{n+1},2^{n+1}} \supset M_1^{k_{n+2},2^{n+2}} \supset \dots \quad \text{with } \omega \in M_1^{k_j,2^j}$$

for all $j \geq N$. It must follow that

$$\frac{k_n t_o}{2^n} \leq \frac{k_{n+1} t_o}{2^{n+1}} \leq \frac{k_{n+2} t_o}{2^{n+2}} \leq \dots \leq \frac{(k_n+1)t_o}{2^n} \quad \text{and}$$

$$\frac{(k_n+1)t_o}{2^n} \geq \frac{(k_{n+1}+1)t_o}{2^{n+1}} \geq \frac{(k_{n+2}+1)t_o}{2^{n+2}} \geq \dots \geq \frac{k_n t_o}{2^n} .$$

Let

$$\lim_{n\to\infty} \frac{k_n t_o}{2^n} = r \quad \text{and} \quad \lim_{n\to\infty} \frac{(k_n+1)t_o}{2^n} = s.$$

Certainly $r \leq s$. If $r < s$, we can find an $\bar{n}$ and $k_{\bar{n}}$ such that

$$r < \frac{k_{\bar{n}} t_o}{2^{\bar{n}}} < \frac{(k_{\bar{n}}+1)t_o}{2^{\bar{n}}} < s$$

which gives a contradiction since $\omega \in M_1^{k_{\bar{n}},2^{\bar{n}}}$. Thus $r = s$.

Therefore $\omega \in A$, implying that $\lim_{n\to\infty} M_1^{2^n} \subseteq A$. Hence $\lim_{n\to\infty} M_1^{2^n} = A$.

From this proof we know that the collection of 1-transition sample paths in the interval $[0,t_o]$, with $x_o(\omega) = 0$, can be expressed in terms of our $\mathcal{M}$-sets. We

can use the same procedure with the 1-transition sample paths beginning with $x_o(\omega) = 1$.

It must be noted that it was just for convenience that we used those $\mathcal{M}$-sets related to the open intervals with dyadic rational endpoints. We could have just as well chosen any other sequence of $\mathcal{M}$-sets whose respective open interval endpoints become dense in $[0,t_o]$. Then in the same fashion, express A in terms of this new sequence. Further, if $\{M_1^{s_n}\}$ denotes the new sequence,

$$\lim_{n\to\infty} M_1^{s_n} = A = \lim_{n\to\infty} M_1^{2^n},$$

implying that the measure that is obtained by using $\{M_1^{s_n}\}$ is equal to the measure obtained by using $\{M_1^{2^n}\}$. We also have that $0 = m_*(A) \leq P(A) \leq \hat{P}(A) = \frac{b}{b-a}(e^{-at_o} - e^{-bt_o})$, since each of the $\mathcal{M}$-sets that express A is bounded below by 0, and above by its separable probability.

For $k > 1$, the k-transition case can be handled similarly. We will show the basic construction for the 2-transition case. We use the following notation:

$$M_{0,j}^{2^n} \equiv \{x_t(\omega)=0 \text{ on } [0,\frac{jt_o}{2^n})\}$$

$$M_{1,j,k}^{2^n} \equiv \{x_t(\omega)=1 \text{ on } (\frac{(j+1)t_o}{2^n},\frac{kt_o}{2^n})\}$$

$$M_{k,0}^{2^n} \equiv \{x_t(\omega)=0 \text{ on } (\frac{(k+1)t_o}{2^n},t_o]\}$$

$$M_2^{j,k,2^n} \equiv M_{0,j}^{2^n} \cap M_{1,j,k}^{2^n} \cap M_{k,0}^{2^n}$$

$$M_2^{2^n} \equiv \bigcup_{j=1}^{2^n-4} \left(\bigcup_{k=j+2}^{2^n-2} M_2^{j,k,2^n} \right)$$

$$A = \{x_t(\omega)=0 \text{ on } [0,\hat{t}); x_t(\omega)=1 \text{ on } (\hat{t},\bar{t});$$

$$x_t(\omega)=0 \text{ on } (\bar{t},t_o] \text{ for some } \hat{t},\bar{t}\}.$$

As before, we show that $\lim M_2^{2^n}$ exists and equals A, which is the collection of all sample paths that make two transitions in the interval $[0,t_o]$. Though we leave out the proof, such a limit does exist and in fact equals A. All finite-transition sample paths can be expressed in a similar manner, and, hence, are measurable with respect to the measure of the $\mathcal{M}$-sets that express them.

3. Arbitrary Assignment of Measures

We have now reached the heart of our work. In working in the non-separable process, the probability of an event may be any value between the inner and outer measure with respect to the sets in S, which is the σ-algebra before we extended to include the $\mathcal{M}$-sets. However, in assigning an arbitrary measure to a set, we have (implicitly) affected measures of other related sets. As an example, consider the set,

$$M_{1,(r,s)} = \{x_t(\omega)=1 \text{ on } (r,s)\}$$

Let $\hat{P}(M)$ denote the separable probability of the set M. Again we are reminded that all probabilities of M-sets, hereafter, are the conditional probabilities. Now suppose that in a particular process we had that,

$$P(M_{1,(r,s)}) = \alpha\hat{P}(M_{1(r,s)}) \quad \text{where } 0 \le \alpha < 1.$$

Take an interval which contains (r,s), say the open interval (c,d). Then we know that $M_{1,(c,d)} \subseteq M_{1(r,s)}$, and

$$P(M_{1,(c,d)}) = P(M_{1,(c,r)})\, P(M_{1,(r,s)})\, P(M_{1,(s,d)})$$

$$= \beta\hat{P}(M_{1,(c,r)})\, \alpha\hat{P}(M_{1,(r,s)})\, \gamma\hat{P}(M_{1,(s,d)})$$

$$\le \alpha\hat{P}(M_{1,(c,d)}), \quad \text{since } \beta,\gamma \in [0,1].$$

Thus, the probability of all subsets of $M_{1,(r,s)}$ have been affected, since in this example $\alpha < 1$. Now consider a subinterval of (r,s), say (e,f). We have that $M_{1,(r,s)} \subseteq M_{1,(e,f)}$, and that

$$P(M_{1,(r,s)}) = P(M_{1,(r,e)})\, P(M_{1,(e,f)})\, P(M_{1,(f,s)})$$

$$= \beta_1\hat{P}(M_{1,(r,e)})\, \alpha_1\hat{P}(M_{1,(e,f)})\, \gamma_1\hat{P}(M_{1,(f,s)})$$

$$= \alpha_1\beta_1\gamma_1 P(M_{1,(r,s)})$$

where α_1, β_1, γ_1 ε [0,1] and $\alpha_1\beta_1\gamma_1 = \alpha < 1$. Since α_1, β_1, and γ_1 cannot all be equal to 1, at least one of the supersets $M_{1,(r,e)}$, $M_{1,(e,f)}$, $M_{1,(f,s)}$ containing $M_{1,(r,s)}$ has a probability less than its separable probability, and its corresponding factor, α_1, β_1, or γ_1, must be greater than or equal to α. Thus, assigning an arbitrary measure to a set may or may not affect the probability of a superset, but at least one superset must have a probability less than its separable probability.

Since we extended our σ-algebra $\mathcal{S}$ to include the $\mathcal{M}$-sets, our first concern in assigning probabilities should be with these sets. Then all other sets generated by these $\mathcal{M}$-sets will automatically be assigned a probability. So we restrict our work to the $\mathcal{M}$-sets hoping to discover some criteria for assigning probabilities in our non-separable process.

From the preceding paragraph we already have two conditions:

1) Altering the probability of a set affects the probability of all subsets.

2) Altering the probability of a set affects the probability of at least one superset.

We have another criterion which is obtained from the manner in which we extended our σ-algebra S. Suppose $r_n \downarrow r$ and $s_n \uparrow s$. Then

$$(11) \qquad \lim_{n\to\infty} M_{1,(r_n,s_n)} = M_{1,(r,s)},$$

which can be easily shown. If $r_n \downarrow r$ and $s_n \uparrow s$ it is false that the $\lim_{n\to\infty} M_{1,(r_n,s_n)} = M_{1,[r,s]}$. We reason this in the following manner: Since $\lim_{n\to\infty} M_{1,(r_n,s_n)} = \bigcup_{n=1}^{\infty} M_{1,(r_n,s_n)}$, to say that $\bigcup_{n=1}^{\infty} M_{1,(r_n,s_n)} = M_{1,(r,s)}$ we mean that for any $\omega \in M_{1,[r,s]}$, we have that $\omega \in M_{1,(r_n,s_n)}$ for some n. If we choose $\omega \in M_{0,[0,r)} \cap M_{1,[r,s]} \cap M_{0,(s,t_o]}$, then certainly $\omega \in M_{1,[r,s]}$ but $\omega \notin M_{1,(r_n,s_n)}$ for all n. Thus, $\lim_{n\to\infty} M_{1,(r_n,s_n)} \neq M_{1,[r,s]}$. However, from (11) we have that,

3) If $r_n \downarrow r$ and $s_n \uparrow s$, then $\lim_{n\to\infty} P(M_{1,(r_n,s_n)}) = P(M_{1,(r,s)})$. We term this third condition our "continuity condition".

Let us consider an example of an assignment of probabilities in a non-separable process, where the assignment is consistent with the three criteria we have established. In a particular non-separable process we will make the assignment of probabilities to our $\mathcal{M}$-sets in the following manner:

$$P(M_{1,(r,s)}) = \begin{cases} \hat{P}(M_{1,(r,s)}) \text{ if } (r,s)\subset[0,t_o/2) \\ 2^{-2(s-r)/t_o}\ \hat{P}(M_{1,(r,s)}) \text{ if } (r,s)\subset(\frac{t_o}{2},t_o) \\ 2^{1-(2s/t_o)}\ \hat{P}(M_{1,(r,s)}) \text{ otherwise.} \end{cases}$$

To those $\mathcal{M}$-sets in which the value is 0 throughout the interval, we use the same assignment of probabilities. It is easily verified that such an assignment is consistent with the three criteria. However, let us examine the process more closely. We introduce the functions α, β, and γ defined by,

$$\alpha((r,s)) = P(M_{1,(r,s)})$$

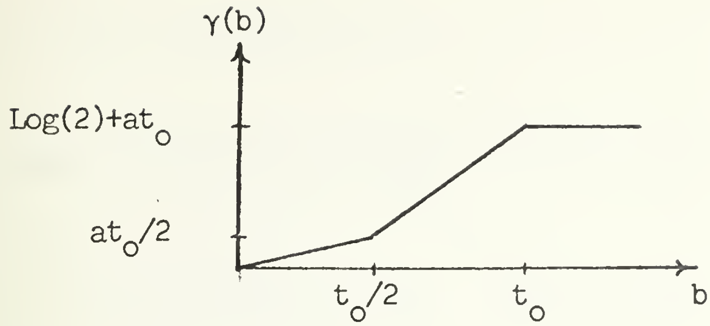
$$\beta((r,s)) = -\log(\alpha((r,s)))$$

$$\gamma(s) = \beta((0,s)).$$

We observe that $\beta((r,s)) = \gamma(s) - \gamma(r)$, and because of the assignment of probabilities, our γ function is more precisely defined by,

$$\gamma(s) = \begin{cases} as & \text{if} \quad s \leq \frac{t_o}{2} \\ 2(s-\frac{t_o}{2})\log(2) + as & \text{if} \quad s \geq \frac{t_o}{2} \end{cases}$$

since $\hat{P}(M_{1,(r,s)}) = e^{-a(s-r)}$, where a is the value obtained at the beginning of this chapter. The graph of the function looks like:



Though this approach worked nicely for this example, the approach will fail when a particular assignment gives probability 0 to some $\mathcal{M}$-set, say $M_{1,I}$, in which case $\beta(I)$ has value ∞, and, hence, γ is not finite. With γ not finite, consistency tests must be handled more delicately.

It is now our interest to look for finer conditions, and we shall do so by looking at our assignments from a different vantage point. Noting that there is a relationship between the probabilities of a set and the probabilities of its subsets and supersets, we may think of the ratio of the separable probability over the non-separable probability of an $\mathcal{M}$-set as a function of the endpoints of the $\mathcal{M}$-set's interval. This ratio is what we will call the factor function $F(\cdot,\cdot)$, and write the probability of an $\mathcal{M}$-set in the following form:

$$P(M_{1,(r,s)}) = F(r,s)\ \hat{P}(M_{1,(r,s)}).$$

Now we wish to characterize the function F. One obvious condition that F must satisfy is,

1) $0 \leq F(r,s) \leq 1.$

We obtain another condition by examining the relationship,

$$F(r,s)\hat{P}(M_{1,(r,s)}) = P(M_{1,(r,s)}) = P(M_{1,(r,t)})P(M_{1,(t,s)})$$

$$= F(r,t)\hat{P}(M_{1,(r,t)})F(t,s)\hat{P}(M_{1,(t,s)})$$

$$= F(r,t)F(t,s)\hat{P}(M_{1,(r,t)})\hat{P}(M_{1,(t,s)})$$

$$= F(r,t)F(t,s)\hat{P}(M_{1,(r,s)})$$

where $r < t < s$. Thus, we have a second condition:

2') $F(r,s) = F(r,t)F(t,s)$ for all $r < t < s$.

Since the interval (r,r) is actually the void set, we know that $P(M_{1,(r,r)}) = P(\Omega) = 1$. So $F(r,r) = 1$ for all r, and we may write condition (2') as

2) $F(r,s) = F(r,t)F(t,s)$ for all $r \leq t \leq s$.

A third condition is derived from criterion (3) stated earlier; we have

3') If $r_n \downarrow r$ and $s_n \uparrow s$, then $\lim_{n\to\infty} F(r_n,s_n) = F(r,s)$.

An equivalent statement of (3') is

3) $F(r,s)$ is right continuous in r, and left continuous in s.

The factor function of the example presented earlier can be shown to satisfy these three conditions.

From conditions (1) and (2), we note that F is non-increasing for increasing s or for decreasing r. A subsequent result is that F is an upper-semicontinuous function, which we will show. Choose $c > 0$, and consider the set $\{(r,s) | F(r,s)<c\} \equiv W$. If $(\overline{r},\overline{s}) \in W$, there exists $\delta > 0$ such that $F(\overline{r}+\delta,\overline{s}-\delta) < c$, by criterion (3). Since F is non-increasing for increasing s or for decreasing r, the northwest region of our domain with respect to the point $(\overline{r}+\delta,\overline{s}-\delta)$ is contained in W. In particular there is an open area in this northwest region containing $(\overline{r},\overline{s})$. Thus, W is open. Therefore, F is an upper-semicontinuous function.

Now let $\mathcal{F} = \{F | F$ satisfies conditions (1), (2), and (3)$\}$. Let F_n be a decreasing sequence in $\mathcal{F}$. Since each F_n is bounded, the sequence converges to some $\overline{F}$. We wish to show that $\overline{F}$ is also in $\mathcal{F}$. Since the sequence is in $\mathcal{F}$, for each n,

$$F_n(r,s) = F_n(r,t)\, F_n(t,s) \quad \text{for all } r \leq t \leq s.$$

we know also that $\lim_{n\to\infty} F_n(r,s) = \overline{F}(r,s)$ and that

$$\lim_{n\to\infty} (F_n(r,t)\ F_n(t,s)) = \overline{F}(r,t)\ \overline{F}(t,s).$$

Thus, $\overline{F}(r,s) = \overline{F}(r,t)\ \overline{F}(t,s)$ for all $r \leq t \leq s$, so condition (2) is satisfied. For condition (3), let $r_k \downarrow r$. Then for all n,

$$\lim_{k\to\infty} F_n(r_k,s) = F_n(r,s).$$

Choose $\varepsilon > 0$. For some $\overline{n}$, $|F_{\overline{n}}(r,s) - \overline{F}(r,s)| < \frac{\varepsilon}{2}$. Now there exists a $\overline{k}$ such that for all $k > \overline{k}$, $|F_{\overline{n}}(r_k,s) - F_{\overline{n}}(r,s)| < \frac{\varepsilon}{2}$. We note that

$$F_{\overline{n}}(r_k,s) \geq \overline{F}(r_k,s) \geq \overline{F}(r,s) \quad \text{for all } k > \overline{k}.$$

Thus, $|\overline{F}(r_k,s) - \overline{F}(r,s)| \leq |F_{\overline{n}}(r_k,s) - \overline{F}(r,s)|$

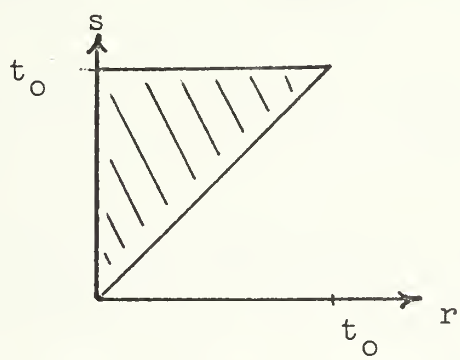
$$\leq |F_{\overline{n}}(r_k,s) - F_{\overline{n}}(r,s)| + |F_{\overline{n}}(r,s) - \overline{F}(r,s)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{for all } k > \overline{k}(\overline{n}(\varepsilon)) = \overline{k}(\varepsilon).$$

Therefore, $\lim_{k\to\infty} F(r_k,s) = \overline{F}(r,s)$, likewise for s, and condition (3) is satisfied. Obviously, condition (1) is satisfied, giving us the desired result that $\overline{F} \in \mathcal{F}$. A

stronger conjecture is that each function in $\mathcal{F}$ is the limit of a decreasing sequence of continuous functions in $\mathcal{F}$. It remains to be proved.

We may picture the domain of the functions in $\mathcal{F}$ by considering r-s coordinate system, where r and s range from 0 to t_o. Since s is never less than r, the domain of each F is the triangular region bounded by the lines $r = 0$, $s = t_o$, and $r = s$. We illustrate:



By the argument used earlier, $F(r,r) = 1$ for all r. Thus, the value of F along the line $r = s$ is 1 for any assignment of probabilities.

Now suppose that for a particular pair, r_o, s_o, $F(r_o, s_o) = 0$. Looking at our coordinate system, all points to the northwest of (r_o, s_o) have value 0 under the function F. By the same reasoning if $F(\bar{r}, \bar{s}) > 0$, then all points to the southeast of $(\bar{r}, \bar{s})$ have values greater than 0 under the function F. We also know that since

$$F(r_o,s_o) = F(r_o,s_o-\delta)\ F(s_o-\delta,s_o) = 0, \text{ then either}$$

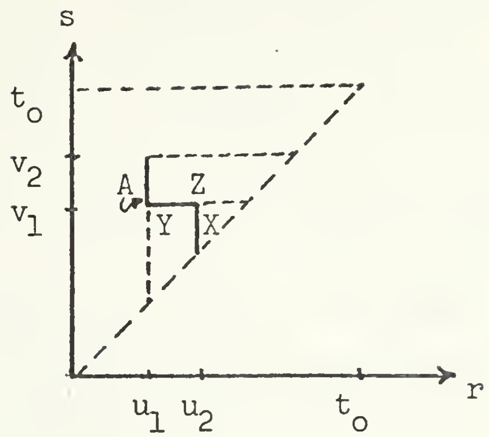
$$F(r_o,s_o-\delta) = 0, \text{ or } F(s_o-\delta,s_o) = 0.$$

Thus, there exists some point of value 0 under F along the two lines which form the boundary of all the points to the southeast of (r_o,s_o). Suppose it is the point $(r_o,s_o-\delta)$. By condition (3), all points between $(r_o,s_o-\delta)$ and (r_o,s_o) have value 0 under F. If we add on an additional property that for all $\varepsilon_1,\varepsilon_2 > 0$, $F(r_o+\varepsilon_1,s_o-\varepsilon_2) > 0$, we can argue that for γ such that $0 < \gamma < \delta$, and all $\varepsilon > 0$,

$$F(r_o,s_o-\delta-\varepsilon)\ F(s_o-\delta-\varepsilon,s_o-\delta+\gamma) = F(r_o,s_o-\delta+\gamma) = 0.$$

Since $F(s_o-\delta-\varepsilon,s_o-\delta+\gamma) > 0$ by hypothesis, $F(r_o,s_o-\delta-\varepsilon) = 0$ for all $\varepsilon > 0$. Therefore with the added property, all points between (r_o,s_o) and (r_o,r_o) have value 0 under F. If $(s_o-\delta,s_o)$ is the point having value 0 under F, instead of $(r_o,s_o-\delta)$, the same arguments would hold for the points between (r_o,s_o) and (s_o,s_o).

Consider the following set of points along a path, A, in the domain of F. From this path we form the regions X, Y, and Z. If we specify arbitrary positive values to all points on path A, and if these values are consistent with the three conditions of F, then we may argue that all points in the regions X, Y, and Z have values determined



by the values along the path A. For example, consider a point, (r,s), in the region X. We have that

$$F(u_2,s) = F(u_2,r)\ F(r,s), \quad \text{by condition (2).}$$

Since we have specified the values for $F(u_2,s)$ and $F(u_2,r)$, $F(r,s)$ is forced to assume a particular value. We may use the same argument for all points along the northern border of X, which is a continuation of the horizontal line in path A. Thus, all points between (u_1,v_1) and (v_1,v_1) have a specified value, and, hence, we may determine the values in the region Y in the same manner as we did in the region X. Now we have values for all points between (u_1,v_2) and (u_1,u_1), and, therefore, we may determine the values in the region Z. From this result we conclude that it is sufficient to define our function F only along a path such

as A in order to define the function in the smallest southeast region containing the path.

From this development of the function F, we conclude our work with a series of results.

Lemma 1: If $F(r,s) > 0$, then

i) $F(r,\cdot)$ is continuous on $[r,s]$

and, ii) $F(\cdot,s)$ is continuous on $[r,s]$

Proof: For i) we know that for $\bar{s} \in [r,s]$, $\lim_{\delta \to 0} F(r,\bar{s}-\delta) = F(r,\bar{s})$. Since $F(\bar{s}+\delta,s) > 0$ for all $\delta > 0$, and since the $\lim_{\delta \to 0} F(\bar{s}+\delta,s) = F(\bar{s},s) > 0$

$$\lim_{\delta \to 0} F(r,\bar{s}+\delta) = \lim_{\delta \to 0} (F(r,s)/F(\bar{s}+\delta,s)) = F(r,\bar{s}).$$

Thus, i) is true, and similarly, ii) is true.

Since F is continuous along the line $r = s$, by Lemma 1, F is continuous on all the boundary of a triangular region in the domain of F. Thus,

Lemma 2: If $F(r,s) > 0$, then F is continuous on the closed, southeastern region of the point (r,s).

The lemma follows since the boundary values determine the interior values.

Theorem: If $\underline{\lim}_{\substack{r\to r_0 \\ s\to s_0}} F(r,s) > 0$, then $\lim_{\substack{r\to r_0 \\ s\to s_0}} F(r,s) = F(r_0,s_0)$.

Proof: If $\underline{\lim}_{\substack{r\to r_0 \\ s\to s_0}} F(r,s) > 0$, then there is a neighborhood about (r_0,s_0) such that F is positive in this region. In particular, F is positive at some point in the neighborhood which is northwest of (r_0,s_0). Thus, by Lemma 2, F is continuous at the point (r_0,s_0).

Corollary: The $\lim_{n\to\infty} F(a-\frac{1}{n},a+\frac{1}{n})$ either equals 0 or 1.

Proof: If for some N, $F(a-\frac{1}{N},a+\frac{1}{N}) > 0$, then by Lemma 2, F is continuous at (a,a). Thus $\lim_{n\to\infty} F(a-\frac{1}{n},a+\frac{1}{n}) = F(a,a) = 1$. If no such N exists, then for all n, $F(a-\frac{1}{n},a+\frac{1}{n}) = 0$. Thus, $\lim_{n\to\infty} F(a-\frac{1}{n},a+\frac{1}{n}) = 0$, and the corollary is proved.

Corollary: The probability of the event $\{\lim_{s\to t} x_s = 1\}$ is either 0 or $P(x_t = 1)$. That is to say, the sample paths passing through 1 at time t are either almost all continuous or almost all discontinuous.

Proof: Consider the set $Q_n \equiv M_{1,(t-\frac{1}{n},t+\frac{1}{n})}$, and let $Q = \bigcup_{n=1}^{\infty} Q_n$. The event Q is not the same as the event $\{x_t = 1\}$. But from the previous corollary, $P(Q) = P(x_t = 1)$, unless the Q_n's are null sets, in which case $P(Q) = 0$.

CONCLUSION

In the non-separable process we have reduced the probability of some (or all) of our $\mathcal{M}$-sets with respect to its separable probability. Since the probability of an $\mathcal{M}$-set cannot be greater than its separable probability, an interesting question is which events in the non-separable process have their probabilities increased when the probability of an $\mathcal{M}$-set is decreased. We do know that such events belong to the class of sets in which an "infinite number of transitions" have taken place over an interval. However, such events have separable probability 0. Since the factor function F - approach seemed to answer many vague ideas about non-separability, perhaps with a fuller characterization of F, such questions as that mentioned above may be answered.

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ABSTRACT

The work investigates the notion of separability in continuous parameter stochastic processes. It explores the implications of relaxing the separability hypothesis. Various numerical results are obtained for a particular example, the 0-1 Markov process.

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